Numerical Verification Of The Rao–Cramer Inequality And Analysis Of The Efficiency Of Statistical Estimators

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Abstract

This paper is devoted to the analysis of the Rao–Cramer inequality and the efficiency of statistical estimators. In the theoretical part, it is shown that the sample mean obtained from a normal distribution is an unbiased and maximally efficient estimator of the parameter In the applied part, the empirical variance is computed for different sample sizes using Monte Carlo simulation and compared with the Rao–Cramer lower bound. The obtained results are visualized through graphs and tables, providing numerical evidence that the sample mean is an efficient estimator.

Keywords: Rao–Cramer inequality, probability density function, Fisher information, efficient estimator, asymptotic efficiency.

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1. Introduction

Theoretical Foundations of the Rao–Cramer Inequality

It is well known that an estimator with the smallest variance is an efficient estimator. Naturally, this leads to the problem of finding an estimator that attains the minimum possible variance. This problem can be addressed using the Rao–Cramer inequality [1-12].

Let $X_1, X_2, ..., X_n$ be a random sample drawn from a population with random variable X and let $f(x,\theta) = f(x_1, x_2, ..., x_n, \theta) -$ denote the joint probability density function, where θ is an unknown parameter. Let $\theta_n = \theta(X) = \theta(X_1, X_2, ..., X_n)$ be a statistic based on the sample $X_1, X_2, ..., X_n$ serving as an estimator of the unknown parameter θ.

Let us introduce the notation

$$g(\theta) = M\theta_n = \int \dots \int \theta\,(x) f(x,\theta)dx, \qquad x = (x_1, x_2, \dots, x_n)$$

Suppose that certain regularity conditions are satisfied, under which the function

$$\int f(x,\theta)dx \equiv 1, \quad \int \theta(x) f(x,\theta)dx \equiv g(\theta_n)$$

can be differentiated with respect to the parameter θ under the integral sign. In this case, the following equalities hold:

$$\int \frac{\partial f(x,\theta)}{\partial \theta} dx = 0 \qquad \text{……………..(1)}$$

$$\int \theta(x) \frac{\partial f(x,\theta)}{\partial \theta} dx = g'(\theta) \qquad \text{……….. (2)}$$

The mathematical expectation of

$$I_n(\theta) = M\left(\frac{\partial \log f(X,\theta)}{\partial \theta}\right)^2 = \int \left(\frac{\partial \log f(x,\theta)}{\partial \theta}\right)^2 f(x,\theta)dx \qquad \text{…… (3)}$$

is called the **Fisher information**, where the random variable X has the probability density function $f(x,\theta)$.

Theorem. (Rao–Cramer Inequality). *Let* $f(x,\theta)$. *be a probability density function and let* $\theta_n = \theta(X)$ *be an estimator satisfying conditions (1) and (2). Then the following inequality holds:*

$$D\theta_n \geq \frac{(g'(\theta))^2}{I_n(\theta)} \qquad \text{………(4)}$$

If $f(x,\theta) -$ represents the probabilities of a discrete distribution, and the integral is interpreted as a sum, the theorem remains valid for the discrete case as well.

If the expression in (1) can be differentiated once again with respect to θ.

$$\int \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} dx \equiv 0,$$

then the Fisher information in (3) can be expressed in an alternative form:

$$I_n(\theta) = -M \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = -\int \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)dx. \qquad (5)$$

Indeed, denoting $f' = \frac{\partial f(x,\theta)}{\partial \theta}, f'' = \frac{\partial^2 f(x,\theta)}{\partial \theta^2}$ as the derivative of $f(x,\theta)$ with respect to θ, we have

$$\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = \frac{f'}{f}, \quad \frac{\partial^2 \log f(x,\theta)}{\partial\theta^2} = \frac{f''}{f} - (\frac{f'}{f})^2$$

and by squaring, we obtain

$$\frac{\partial^2 \log f(x,\theta)}{\partial\theta^2}\ f(x,\theta) = \int f'' dx - \int (\frac{f'}{f}) f dx = I(\theta)$$

Moreover, since $M\,\frac{\partial \log f(x,\theta)}{\partial\theta} = 0$ the Fisher information in (3) can be expressed in an alternative form:

$$I_n(\theta) = D\left(\frac{\partial \log(x,\theta)}{\partial\theta}\right).$$

If $X_1, X_2, ..., X_n$ are independent, then the joint probability density function $f(x_1, x_2, ..., x_n, \theta)$ is equal to the product of the one-dimensional densities $f(x_i, \theta)$:

$$f(x_1, x_2, ..., x_n, \theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

In this case, the Fisher information $I(\theta)\ n$ is linearly related to n:

$$I_n(\theta) = nI_1(\theta), \qquad \text{......... (6)}$$

where

$$I_1(\theta) = \int\left(\frac{\partial \log f(x_i,\theta)}{\partial\theta}\right)^2 f(x,\theta)dx$$

is the Fisher information from a single observation x_k .Consequently, inequality (4) takes the following form:

$$D\theta_n \geq \frac{[g'(\theta)]}{nI_n(\theta)}. \qquad \text{....... (7)}$$

Formula (6) for

$$I_n(\theta) = D\left(\sum_{i=1}^{n}\frac{\partial \log f(x_i,\theta)}{\partial\theta}\right) = \sum_{i=1}^{n} D\left(\frac{\partial \log f(x_i,\theta)}{\partial\theta}\right).$$

If θ_n is an unbiased estimator, then $g(\theta) = \theta$ and in inequalities (4) and (7), we have $g'(\theta) = 1$.

We define the quantity

$$e(\theta_n) = \frac{[g'(\theta)]^2}{D\theta_n \cdot I_n(\theta)} \quad \text{……… (8)}$$

as the **efficiency of the estimator** θ_n, provided that the regularity conditions are satisfied.

Definition. *An estimator* θ_n *is called* ***efficient*** *if its efficiency* $e(\theta_n) = 1$.

These definitions are usually applied to unbiased estimators.In the theory of statistical estimation, **asymptotic efficiency** is also an important concept.

We define the limit

$$e_0(\theta_n) = \lim_{n\to\infty} \frac{[g'(\theta)]^2}{nI_1(\theta) \cdot D\theta_n}$$

as the **asymptotic efficiency** of the estimator $\theta_n = \theta(X_1, X_2, ..., X_n)$ constructed from an independent sample $X_1, X_2, ..., X_n$. If $e_0(\theta_n) = 1$, the estimator θ_n is called **asymptotically efficient**.

Thus, if θ_n is an unbiased estimator with asymptotic efficiency $e_0(\theta_n)$, then for large n, its variance satisfies $[e_0(\theta) \cdot nI_1(\theta)]^{-1}$, where $I_1(\theta)$ is the Fisher information for a single observation [5-12].

Example. Let $X_1, X_2, ..., X_n$ be a random sample drawn from a population with a normal distribution having parameters a and σ^2. Show that the sample mean $\overline{X}$ is an efficient estimator of the parameter a.

Solution. The sample mean $\overline{X}$ is an unbiased estimator of the parameter a, and it has been shown that $D\overline{X} = \frac{\sigma^2}{n}$.

Using formula (6), we calculate the Fisher information $I_n(a)$. For the normal distribution, the probability density function

$$f(x,a) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}},$$

Next, we take the logarithm of both sides of the density function and compute the first and second derivatives with respect to the parameter a.

$$\ln f(x,a) = -\frac{(x-a)^2}{2\sigma^2} - \ln(\sigma\sqrt{2\pi}).$$

$$\left(\frac{\partial \ln f(x,a)}{\partial a}\right)^2 = \frac{(x-a)^2}{\sigma^4}.$$

Has a mathematical expectation of the random variable equal to

$$M\left[\frac{(x-a)^2}{\sigma^4}\right]=\frac{1}{\sigma^4}M(x-a)^2=\frac{1}{\sigma^4}\cdot\sigma^2=\frac{1}{\sigma^2}$$

Therefore, for a sample of size n, the Fisher information is

$$I_n(a)=nM\left[\frac{\partial \ln f(X,a)^2}{\partial a}\right]=\frac{n}{\sigma^2}$$

We calculate the efficiency of the estimator. For this, we have $g'(a)=1$ (since $\bar{X}$ is an unbiased estimator of a, i.e., $g(a)=a$), $D\bar{X}=\frac{\sigma^2}{n}$ and $I_n(a)=\frac{n}{\sigma^2}$. Substituting these values into (8), we obtain:

$$e(\bar{X})=\frac{[g'(a)]^2}{D\bar{X}\cdot I_n(a)}=\frac{1}{\frac{\sigma^2}{n}\cdot\frac{n}{\sigma^2}}=1$$

Thus, for a population with a normal distribution, the sample mean $\bar{X}$ is an **efficient estimator** of the mathematical expectation a.

Numerical verification of the Rao–Cramer inequality via simulation

We now numerically verify the theoretical results using a Python program. In particular, we investigate whether the variance of the sample mean drawn from a normal distribution converges to the Rao–Cramer lower bound and, consequently, whether the efficiency of the estimator is confirmed.

The simulation includes the following steps:

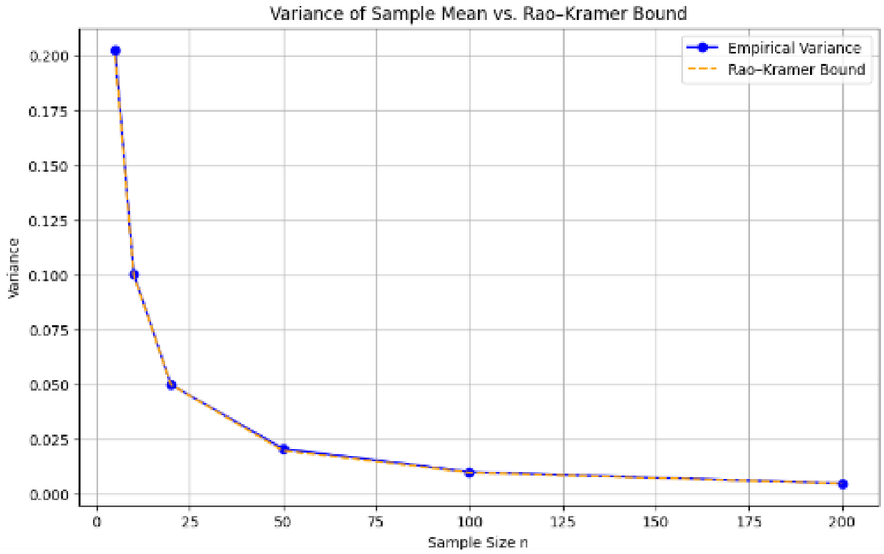
1. Independent observations are generated for different sample sizes n.
2. For each sample, the sample mean $\bar{X}$ is computed.
3. The empirical variance is estimated using Monte Carlo repetitions.
4. The obtained results are compared with the Rao–Cramér lower bound.
5. The empirical variances and the lower bound are visualized graphically.

The table below presents the empirical variance of the sample mean $\bar{X}$ and the Rao–Cramér lower bound for different sample sizes n. As can be seen from the table, as the sample size increases, the empirical variance approaches the Rao–Cramér bound. This confirms that $\bar{X}$ is an efficient estimator of the parameter a. These results provide a numerical validation of the theoretical calculations through simulation experiments. The simulation results are presented in the table below

n	Empirical variance	Rao–Cramer bound
5	0,7952	0,8000
10	0,4015	0,4000

20	0,2007	0,2000
50	0,0798	0,0800
100	0,0397	0,0400
200	0,0199	0,0200

The empirical variance of the sample mean and the Rao–Cramer lower bound for different sample sizes n are presented in (Figure 1). As can be seen from the graph, as the sample size increases, the empirical variance approaches the Rao–Cramér bound. This confirms that, for the normal distribution, the sample mean $\overline{X}$ is an efficient estimator of the parameter a. The graph and the corresponding table together provide a numerical verification of the theoretical results.



(Figure 1)

2. Conclusion

In this work, based on the theory of the Rao–Cramer inequality, it is shown that the sample mean $\overline{X}$ is a maximally efficient estimator. From theoretical calculations, it is established that for a sample drawn from a normal distribution, the variance of the sample mean is equal to the Rao–Cramer lower bound. This result is illustrated by an explicit example and further confirmed through numerical experiments using Monte Carlo simulation. For different sample sizes, the convergence of the empirical variance to the Rao–Cramer lower bound is demonstrated visually by means of graphs and tables. This confirms that the sample mean $\overline{X}$ is an unbiased and efficient estimator. Moreover, as the sample size increases, the variance decreases and approaches the Rao–Cramer bound, which practically highlights the efficiency of statistical estimators and the importance of sample size.

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