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Applications Of Matrix Function And Group Action

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ABSTRACT

The article presents some applications of the matrix function and group action to reveal some properties of matrixes and p-groups.

KEYWORDS

Matrix function, prime number, equivalent relation, p-groups.

INTRODUCTION

Let $A \in M_n(\mathbf{R})$ be a fixed nonzero matrix. Let's define the function $f_A : M_n(\mathbf{R}) \to M_n(\mathbf{R})$ s.t $f_A(X) = AX - XA, \forall X \in M_n(\mathbf{R})$

Now we will examine the some properties of this function:

1-property: $f_A \equiv 0 \Leftrightarrow A = \lambda I_n$, for some $\lambda \in \mathbf{R}$

Proof: $(\Leftarrow) \lambda I_n X = X \lambda I_n = \lambda X$ for every $X \in M_n(\mathbf{R})$

$(\Rightarrow) AX = XA, \forall X \in M_n(\mathbf{R})$ (*), so it is true for $X = A^T \Rightarrow AA^T = A^T A \Rightarrow A -$ normal matrix.

Therefore there exist P-invertible and C-diagonal matrix s.t $A = PCP^{-1}, C = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

By setting this into (*), we get $PCP^{-1}X = XPCP^{-1}$. Now take $X = P^{-1}$, so $PCP^{-1}P^{-1} = P^{-1}PCP^{-1}$

$PCP^{-1} = C \Rightarrow PC = CP \Rightarrow A = PCP^{-1} = CPP^{-1} = C = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$. So, we have found A-diagonal matrix. We have the condition $AX = XA, \forall X \in M_n(\mathbf{R})$ for $A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

Let $X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \Rightarrow AX = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_1 x_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_n x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix}$

$XA = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_n x_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_1 x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix} \Rightarrow \lambda_i x_{1i} = \lambda_1 x_{1i}, i = \overline{1,n}$. We take such X s.t $x_{1i} \neq, i = \overline{1,n} \Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_n \Rightarrow A = \lambda_1 I_n$

2-property: $f_A \circ f_B = f_B \circ f_A \Leftrightarrow AB = BA$

Proof: $(f_A \circ f_B)(X) = f_A(BX - XB) = A(BX - XB) - (BX - XB)A = ABX - AXB - BXA + XBA$

$(f_B \circ f_A)(X) = f_B(AX - XA) = B(AX - XA) - (AX - XA)B = BAX - BXA - AXB + XAB$

According to these $f_A \circ f_B = f_B \circ f_A \Leftrightarrow ABX - AXB - BXA + XBA = BAX - BXA - AXB + XAB, \forall X \in M_n(\mathbf{R})$

$\Leftrightarrow (AB - BA)X = X(AB - BA), \forall X \in M_n(\mathbf{R}) \Leftrightarrow f_{AB-BA} \equiv 0 \Leftrightarrow AB - BA = 0$ or $AB - BA = \lambda I_n$, but second case never happens . Because $0 = tr(AB - BA) = tr(\lambda I_n) = \lambda n \neq 0$. Thus $AB = BA$. $f_A \circ f_B = f_B \circ f_A \Leftrightarrow AB = BA$

3-property: If A is a matrix with n distinct eigenvalues, then the dimension of $\ker(f_A)$ is equal to n.

Proof: A has n distinct eigenvalues, then the Jordan normal form of A is $\underbrace{\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}}_{C} \Rightarrow A = PCP^{-1}$

, $\ker(f_A) = \{X : f_A(X) = 0 \Leftrightarrow AX = XA\}$

$$AX = XA \Leftrightarrow PCP^{-1}X = XPCP^{-1} \Leftrightarrow P^{-1}(PCP^{-1}X)P = P^{-1}(XPCP^{-1})P \Leftrightarrow C(P^{-1}XP) = (P^{-1}XP)C$$

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$\Leftrightarrow P^{-1}XP \in \ker(f_C)$. Therefore $\ker(f_A) \cong \ker(f_C)$, in that $X \to P^{-1}XP, P^{-1}YP \leftarrow Y$. They are linear and their compositions are equal to identity. So, it is enough to find $\dim(\ker(f_C))$.

$$C = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow \ker(f_C) = \{X : CX = XC\}, X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

$$CX = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_1 x_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_n x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix} = XC = \begin{pmatrix} \lambda_1 x_{11} & \cdots & \lambda_n x_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_1 x_{n1} & \cdots & \lambda_n x_{nn} \end{pmatrix} \Leftrightarrow \lambda_i x_{ij} = \lambda_j x_{ij}$$

$$\Leftrightarrow (\lambda_i - \lambda_j)x_{ij} = 0 \Leftrightarrow x_{11}, x_{22}, ..., x_{nn} - \forall, x_{ij} = 0, i \neq j$$

$$\ker(f_C) = \{X : X = \begin{pmatrix} x_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_{nn} \end{pmatrix} = x_{11}\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + ... + x_{nn}\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}\}$$

$$= span\{\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, ..., \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}\} \Rightarrow \dim(\ker(f_C)) = n = \dim(\ker(f_A))$$

Problem-1: Let G be a p-group, i.e $|G|$ is a degree of p, where p is a prime number. $\cdot : G \times X \to X$ be a left-action on X and $F \subset X$ be a subset s.t $F := \{x \in X : g \cdot x = g, \forall g \in G\}$, then $|X| - |F|$ is divisible by p, where $|Y|$ denotes the number of elements of Y.

Proof using the theorem: We build equivalent relation s.t $a \sim b \overset{def}{\Leftrightarrow} a = gb, \exists b \in G$. This relation divides X to orbits, i.e $X = \bigcup_{i=1}^{k} [a_k]$.

Lemma: $\forall a \in X, G_a = \{g \in G \mid ga = a\}$ is a subgroup of G and $|G : G_a| = |[a]|$.

For $x_0 \in F, [x_0] = \{x_0\}$, because $[x_0] = \{gx_0 = x_0 \mid g \in G\} = \{x_0\}$

$$\Rightarrow X = (\bigcup_{x \in F} \{x\}) \cup (\bigcup_{x \notin F} [x]) \Rightarrow X / F = \bigcup_{x \notin F} [x] \Rightarrow |X / F| = |X| - |F| = \sum_{x \notin F} |[x]|$$

$\left| [x] \right|_{x \notin F} = |G : G_x| = |G| : |G_x| = p^{\alpha} : p^{k} = p^{\alpha - k}, \alpha \neq k$ because, if $\alpha = k$ then $G = G_x \Rightarrow \forall g \in G, gx = x \Rightarrow x \in F$, but we are looking at $x \notin F \Rightarrow |X| - |F| \vdots p$

Problem-2: Let G be a p-group, i.e $|G|$ is a degree of p, where p is a prime number. If H is a nontrivial normal subgroup of G, then prove that $H \cap Z(G) \neq \{e\}$, where $Z(G) := \{y \in G : \forall x \in G, xy = yx\}$ **center** and e is identity element of G.

Proof: We construct left-action on H as $\cdot : G \times H \to H, g \cdot h \to g^{-1}hg$. Then we set relation on H s.t $a \sim b \overset{def}{\Leftrightarrow} a = gb = g^{-1}bg, \exists b \in G \Leftrightarrow ga = bg, \exists b \in G$. This relation divides H into orbits, i.e $H = \bigcup_{i=1}^{k} [a_k] = \sum_{i=1}^{k} |G : G_a|$. Suppose $H \cap Z(G) = \{e\} \Rightarrow \forall h \neq e, \exists g_0 : g_0 h \neq hg_0$

If we denote $G_h = \{g : gh = hg\}$, then for $h \neq e$ we have $G_h \neq G$ and $G_e = G$.

$\Rightarrow |G : G_h| = |G| : |G_h| = p^n : p^s = p^{n-s}, n-k \geq 1$ and $|G : G_e| = |G : G| = 1 \Rightarrow |H| = \sum_{i=1}^{k} |G : G_a| = \sum_{i=1}^{k} p^{n-s_i} + 1 \equiv 1 (\mathrm{mod}\ p)$. But H is a subgroup of G and because of this $p^n = |G| \vdots |H| \Rightarrow |H| = p^m \equiv 1 (\mathrm{mod}\ p) \Rightarrow m = 0$ this means H must be trivial subgroup. So, our assumption is not correct. $\Rightarrow H \cap Z(G) \neq \{e\}$.

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