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About One Theorem Of 2x2 Jordan Blocks Matrix

Hojiyev Dilmurodjon Bahodirovich
Math Teacher Of The Lyceum Of Andijan State University, Uzbekistan

Muhammadjonov Akbarshoh Akramjon Og`Li
The Third-Year Student Of Andijan State University, Uzbekistan

Muzaffarova Dilshoda Botirjon Qizi
The First-Year Student Of Andijan State University, Uzbekistan

Ibrohimjonov Islombek Ilhomjon O`G`Li
The First-Year Student Of Andijan State University, Uzbekistan

Ahmadjonova Musharrafxon Dilmurod Qizi
The First-Year Student Of Andijan State University, Uzbekistan

ABSTRACT

In this paper, we have studied one theorem on 2x2 Jordan blocks matrix. There are 4 important statements which is used for proving other theorems such as in the differensial equations. In proving these statements, we have used mathematic induction, norm of matrix, Taylor series of $f(x) = e^X$.

KEYWORDS

Matrix, vector, mathematic induction method, equality, inequality, norm, expansion of series, estimation.

INTRODUCTION

First of all, we present the Theorem that is the main result of our article, then we prove statements one after another in the Theorem.

Theorem. If

$$A_i = \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix}, \qquad \lambda_i \geq 0 \quad i = 1,2,\ldots \quad t\epsilon[0;T]$$

then the following statements are true:

1) $e^{A_i t} = e^{-\lambda_i t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}, \qquad i = 1,2,\dots$

2) $e^{A_i(t+h)} = e^{A_i t}\cdot e^{A_i h}, \ \ h\epsilon R.$

3) $\|e^{A_i t}\| \le 1 + T$

4) $\|e^{A_i t} - E_2\| \le t, \qquad E_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$

where T is sufficiently large fixed number, $\|A\| = \max_{|x|=1}|Ax|$.

Proof. We would prefer to present proofs in order which is shown above, respectively.

1. Here, the following Taylor expansion of $f(x) = e^x$ function is important to prove 1) equality, so we use this

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad (1)$$

If we mean $x = A_i t$, then we get a result that:

$$e^{A_i t} = E_2 + A_i t + \frac{(A_i t)^2}{2!} + \frac{(A_i t)^3}{3!} + \dots + \frac{(A_i t)^n}{n!} + \dots = \sum_{n=0}^{\infty}\frac{(A_i t)^n}{n!} \qquad (2)$$

We establish one by one A_i^n $(n = 2,3,\dots)$ in (2) equality for convenience:

$$A_i^{\,2} = \begin{bmatrix}-\lambda_i & 1\\ 0 & -\lambda_i\end{bmatrix}\cdot\begin{bmatrix}-\lambda_i & 1\\ 0 & -\lambda_i\end{bmatrix} = \begin{bmatrix}\lambda_i^{\,2} & -2\lambda_i\\ 0 & \lambda_i^{\,2}\end{bmatrix}$$

$$A_i^{\,3} = \begin{bmatrix}\lambda_i^{\,2} & -2\lambda_i\\ 0 & \lambda_i^{\,2}\end{bmatrix}\cdot\begin{bmatrix}-\lambda_i & 1\\ 0 & -\lambda_i\end{bmatrix} = \begin{bmatrix}-\lambda_i^{\,3} & 3\lambda_i^{\,2}\\ 0 & -\lambda_i^{\,3}\end{bmatrix}$$

$$A_i^{\,4} = \begin{bmatrix}-\lambda_i^{\,3} & 3\lambda_i^{\,2}\\ 0 & -\lambda_i^{\,3}\end{bmatrix}\cdot\begin{bmatrix}-\lambda_i & 1\\ 0 & -\lambda_i\end{bmatrix} = \begin{bmatrix}\lambda_i^{\,4} & -4\lambda_i^{\,3}\\ 0 & \lambda_i^{\,4}\end{bmatrix}$$

As a result of continue doing this process, we obtain:

$$A_i^{\,n} = \begin{bmatrix}(-1)^n\lambda_i^{\,n} & (-1)^{n-1}\cdot n\cdot\lambda_i^{\,n-1}\\ 0 & (-1)^n\cdot\lambda_i^{\,n}\end{bmatrix} \qquad (3)$$

Now, we indicate that (3) equality which we mentioned for ${A_i}^n$ by inductive method is correct for any $n \in N$ by using mathematic induction method so we prove that

$$ {A_i}^{n+1} = \begin{bmatrix} (-1)^{n+1}{\lambda_i}^{n+1} & (-1)^n \cdot (n+1) \cdot {\lambda_i}^n \\ 0 & (-1)^{n+1} \cdot {\lambda_i}^{n+1} \end{bmatrix} $$

$$ A_i^{n+1} = A_i^n \cdot A_i = $$

$$ = \begin{bmatrix} (-\lambda_i)^n & n \cdot (-\lambda_i)^{n-1} \\ 0 & (-\lambda_i)^n \end{bmatrix} \cdot \begin{bmatrix} -\lambda_i & 1 \\ 0 & -\lambda_i \end{bmatrix} = $$

$$ = \begin{bmatrix} (-1)^{n+1}{\lambda_i}^{n+1} & (-1)^n \cdot (n+1) \cdot {\lambda_i}^n \\ 0 & (-1)^{n+1} \cdot {\lambda_i}^{n+1} \end{bmatrix} $$

This indicates that (3) equality is correct for any $n \in N$.

Therefore,

$$ e^{A_i t} = \sum_{n=0}^{\infty} \frac{(A_i \cdot t)^n}{n!} = \sum_{n=0}^{\infty} \frac{A_i^n \cdot t^n}{n!} = $$

$$ \sum_{n=0}^{\infty} \begin{bmatrix} \frac{(-\lambda_i t)^n}{n!} & \frac{(-\lambda_i t)^{n-1}}{(n-1)!} \cdot t \\ 0 & \frac{(-\lambda_i t)^n}{n!} \end{bmatrix} = e^{-\lambda_i} \cdot \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} $$

1. Next, we indicate that 2) equality is right.

If we mean $x = A_i(t+h)$ and close to the growth of (1) we can write the left hand of 2) equality, we get a result

$$ e^{A_i(t+h)} = E_3 + A_i(t+h) + \frac{[A_i(t+h)]^2}{2!} + \frac{[A_i(t+h)]^3}{3!} + \ldots $$

$$ + \ldots + \frac{[A_i(t+h)]^n}{n!} + \ldots += E_2 + A_i(t+h) + A_i^2\left(\frac{t^2}{2!} + t \cdot h + \frac{h^2}{2!}\right) + \ldots $$

$$+\ldots+A_i^n\left[\frac{t^n}{n!}+\frac{t^{n-1}}{n-1}\cdot\frac{h}{1!}+\ldots+\frac{t^{n-k}}{(n-k)!}\cdot\frac{h^k}{k!}+\ldots+\frac{h^n}{n!}\right]+\ldots=$$

$$=\left(E_2+A_it+\frac{(A_it)^2}{2!}+\ldots+\frac{(A_it)^n}{n!}+\ldots\right)\left(E_2+A_ih+\frac{(A_ih)^2}{2!}+\ldots+\frac{(A_ih)^n}{n!}+\ldots\right)$$

$$=e^{A_it}\cdot e^{A_ih}$$

Proof is completed.

2. Next, we prove 3) inequality

In fact, we get a result from 1) that

$$\left\|e^{A_it}\right\|=\left\|e^{-\lambda_i}\cdot\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\right\|\le\left\|\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\right\|$$

According to the original estimate $\|A\|=\max_{|x|=1}|Ax|$ above:

$$\left\|e^{A_it}\right\|\le\left\|\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\right\|=\max_{|x|=1}\left\|\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right\|=$$

$$\max_{|x|=1}\left\|\begin{bmatrix}x_1+t\cdot x_2\\ x_2\end{bmatrix}\right\|$$

where $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $|x|=\sqrt{x_1^2+x_2^2}=1.$

We mean a, b, c vectors by

$$a=\begin{bmatrix}x_1\\ x_2\end{bmatrix}, b=\begin{bmatrix}t\cdot x_2\\ 0\end{bmatrix}, c=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}x_1+t\cdot x_2\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}t\cdot x_2\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}=a+b+c$$

$$\left\|e^{A_it}\right\|\le\max_{|x|=1}|a+b+c|$$

We know that $|a+b+c|\le|a|+|b|+|c|$ inequality is correct for any a, b, c vectors. By using this,

$$\left\|e^{A_i t}\right\| \leq \max_{|x|=1}|a+b+c| \leq |a|+|b|+|c| =$$

$$= \sqrt{{x_1}^2+{x_2}^2}+\sqrt{t^2 \cdot {x_2}^2} =$$

$$= 1+t \cdot \sqrt{{x_2}^2} \leq 1+t$$

because of $t \in [0; T]$, we get a result

$$\left\|e^{A_i t}\right\| \leq 1+T$$

3. Eventually, we move to prove 4) inequality in the **Theorem**:

$$\left\|e^{A_i t}-E_2\right\| \leq t$$

where $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We have the following expressions by using 1) equality and $e^{-\lambda_i t} \leq 1$

inequality:

$$\left\|e^{A_i t}-E_2\right\| = \left\|e^{-\lambda_i t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\| \leq$$

$$\leq \left\| \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\| = \left\|\begin{matrix} 0 & t \\ 0 & 0 \end{matrix}\right\| =$$

$$= \max_{|x|=1}\left|\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right| = \max_{|x|=1}\left|\begin{bmatrix} t \cdot x_2 \\ 0 \end{bmatrix}\right|$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $|x| = \sqrt{x_1^2+x_2^2} = 1$.

Close to the proof of the third property we mean a, b, c vectors by

$$a = \begin{bmatrix} t \cdot x_2 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And according to $|a+b+c| \leq |a|+|b|+|c|$ inequality

$$\left\|e^{A_i t} - E_2\right\| \leq \max_{|x|=1}|a + b + c| \leq [|a| + |b| + |c|]_{|x|=1} =$$

$$\sqrt{t^2 \cdot {x_2}^2} = t \cdot \sqrt{{x_2}^2} \leq t.$$

This **Theorem** is proved completely.

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